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Non-commutative geometry and the Ising model

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Abstract. The main aim of this work is to present an interpretation of Ising-type models as a kind of field theory in the framework of non-commutative geometry. We present the method and construct sample models of field theory on discrete spaces using the tools of discrete geometry introduced. We write the action for a few models, then we compare them with various models of statistical physics. We also construct the gauge theory with a discrete gauge group.

1. Introduction

Non-commutative geometry can be considered as a set of mathematical tools which, applied to theoretical physics, can significantly improve and enlarge the possibilities of model-building in field theory [1–4]. These methods allow us to apply the instruments of differential geometry not only to manifolds but also to many non-standard objects like discrete spaces and quantum spaces. For instance, it appears that the standard model of electroweak interactions can be properly described by the product of continuous and discrete geometry [2–6], thus suggesting the significance of the role of discrete spaces in physics.

In our earlier work [8] we constructed the necessary tools to build sample models in the framework of discrete geometry and we used them to construct gauge theories. Now, we want to turn our attention to the already existing class of physical models, which are also situated on discrete spaces, the best known example being the Ising model [9]. The problem that we wish to consider in this work is whether such models can be reformulated as a field theory constructed according to the rules of non-commutative geometry. Our attempt is to show the general procedure of such a construction and to illustrate it with simple examples. We also include a brief account of the differential calculus and metric properties of discrete spaces.

2. Differential calculus

This section is devoted to a brief review of differential calculus on discrete spaces. We state here only the most important facts and results; details can be found in our earlier work [8].

Let G be a finite group and \mathcal{A} be the algebra of functions on G , which are valued in a field \mathcal{F} . The natural choice for \mathcal{F} is the field of complex numbers \mathbb{C} , however, one may also consider other possibilities. We denote the group multiplication by \odot and the size of

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the group by N_G . The right and left multiplications on G induce natural automorphisms of \mathcal{A} , R_g and L_g , respectively

$$(R_h f)(g) = f(g \odot h) \quad (1)$$

with a similar definition for L_g .

Now we shall construct the extension of \mathcal{A} into a graded differential algebra. First we introduce the space of one-forms as a free left-module over \mathcal{A} , which is generated by the elements χ^g , $g \in G'$, where by G' we denote $G \setminus \{e\}$. Then we define the external derivative d on the zero-forms (elements of \mathcal{A}) in the following way:

$$da = \sum_g (a - R_g(a)) \chi^g. \quad (2)$$

The external derivative is nilpotent and obeys the Leibniz rule, provided that the module of one-forms simultaneously has the structure of a right-module, as defined for its generators:

$$\chi^g a = R_g(a) \chi^g \quad a \in \mathcal{A} \quad g \in G' \quad (3)$$

and the action of d on the generators χ^g is as follows:

$$d\chi^g = - \sum_{h,k} C_{hk}^g \chi^h \otimes \chi^k \quad g \in G' \quad (4)$$

where the constants C_{hk}^g are the structure constants, obtained from the relations

$$(1 - R_i)(1 - R_j) = \sum_k C_{ij}^k (1 - R_k). \quad (5)$$

In the case of the discrete group G their form is rather simple

$$C_{-hk}^g = \delta_h^g + \delta_k^g - \delta_{(k \odot h)}^g. \quad (6)$$

As can already be seen in the formula (4) the higher-order forms are the tensor products of the lower-order ones. Then, the external derivative acts on them in accordance with the graded Leibniz rule

$$d(v \otimes w) = (dv) \otimes w + (-1)^{\deg v} v \otimes (dw). \quad (7)$$

The conjugation in the algebra of forms is taken to be the internal conjugation within the algebra \mathcal{A} for the zero-forms. For higher-order forms it is sufficient to define this operation for the generating one-forms

$$(\chi^g)^* = -\chi^{(g^{-1})}. \quad (8)$$

All these rules give us the structure of the infinite-dimensional differential algebra over the algebra \mathcal{A} . We shall use them as tools to define simple examples of field theories.

3. Metric on discrete spaces

In this section we shall briefly outline the general scheme of the construction and the properties of the metric. We shall give the definitions as well as the intuitive picture.

We define the metric on the module of one-forms, as a middle-linear, \mathcal{A} -valued functional η

$$\eta(a\omega_1, \omega_2 b) = a\eta(\omega_1, \omega_2)b \quad (9)$$

$$\eta(\omega_1 a, \omega_2) = \eta(\omega_1, a\omega_2) \quad (10)$$

This definition is suitable only for the case considered and it has to be modified for other algebras. Both conditions are straightforward generalizations of linearity requirements for the bimodules. In the case of discrete geometry, with the module of one-forms generated by the forms χ^g , the metric is completely determined by its values on the generators,

$\eta^{gh} = \eta(\chi^g, \chi^h)$. Now, because of (10) and the rules of the differential calculus (2)–(4) we obtain the result that η^{gh} must vanish unless $g = h^{-1}$. This means that our metric has only $N_G - 1$ independent components, which we shall denote as E_g

$$\eta^{gh} = E_g \delta^{g(h^{-1})} \quad E_g \in \mathcal{A} \quad g \in G'. \tag{11}$$

If we require that the constructed metric gives rise to a semi-norm, we should restrict ourselves to metrics that are positive definite. For the algebra of \mathbb{C} -valued functions this is equivalent to the choice of real, non-negative E_g . The last issue we wish to highlight is the question of degeneracy. We say that the metric is non-degenerate if the condition $\eta(a^*, a) = 0$ implies that $a = 0$.

The question we wish to raise next is whether this formal construction of the metric can be translated to the metric properties of our base space, i.e. the group G . It is important in the construction of physical theories that we should be able to have the picture of the underlying base space rather than only of the algebra \mathcal{A} . Therefore, we would like to have the possibility of introducing both distances and the concept of nearest neighbours.

We use the following definition for the distance $d(p, q)$ between two points p and q of the base space [4]:

$$d(p, q) = \sup_{\eta(da, da^*) \leq 1} |p(a) - q(a)|. \tag{12}$$

We have identified the base space as the space of characters on the algebra \mathcal{A} , so that the definition (12) makes sense for arbitrary \mathcal{A} . In our case, of course, $p(a) \equiv a(p)$. The inequality $\eta(da, da^*) \leq 1$ means that the function on its left-hand side is maximized by the constant function 1.

Before we present a few simple examples let us observe some general properties of the metric. First, the metric does not have to be symmetric, i.e. $\eta(u, v) \neq \eta(v, u)$. However, after integrating out the result using the Haar integration on G we recover the symmetry.

The distances are, by definition, positive numbers from the field \mathcal{F} , so in our case, where $\mathcal{F} = \mathbb{C}$, they are real positive numbers. Of course, the definition (12) implies the triangle inequality

$$d(p, q) \leq d(p, r) + d(r, q) \tag{13}$$

for any p, q, r .

Finally, let us introduce the notion of the nearest neighbours of a point h , which shall be all such elements of the base space of the form $h \circ g, h \circ (g^{-1})$, where $g \in G'$ and $E_g \neq 0$.

Now, let us proceed with the examples.

\mathbb{Z} with a trivial metric. Let us take the functions E_g determining the metric η to be zero for $g \neq 1$ and $E_1 = 1$. Then, the condition $\eta(da, da^*) \leq 1$ simplifies to $(a(p) - a(p+1))^2 \leq 1$, and one easily finds the distance between $n, m \in \mathbb{Z}$

$$d(n, m) = |n - m|.$$

We can now draw a picture representing this base space. If we connect the nearest neighbours with a link, then each element has two nearest neighbours at the distance 1 and we obtain the image shown in figure 1, which is the most natural representation for \mathbb{Z} .



Figure 1. \mathbb{Z} with a trivial metric.

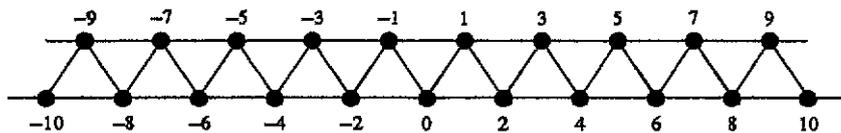


Figure 2. \mathbb{Z} with a non-trivial metric.

\mathbb{Z} with a non-trivial metric. Let us take $E_1 = 1$ and $E_2 = 1$ with all other E_n vanishing. The condition $\eta(da, da^*) \leq 1$ takes now the form

$$((a(n) - a(n + 1))^2 + (a(n) - a(n + 2))^2 \leq 1$$

and we see that the distances are different from those in the previous example. The general formula is rather complicated; we shall only mention that $d(n, n + 1) = d(n, n + 2) = 1$, $d(n, n + 3) = (1/\sqrt{2}) + 1$ and for large m the distance $d(n, n + m)$ is proportional to $(1/\sqrt{5})m$. This result is presented in figure 2, where we see that each point has now four nearest neighbours.

We can proceed further with more sophisticated choices of the metric η^{gh} , deriving in each case the corresponding pictorial representation. Of course, we do not have to deal with infinite groups: one may also take \mathbb{Z}_N , in which case the resulting diagram will be similar, though, of course, it would have the topology of a circle.

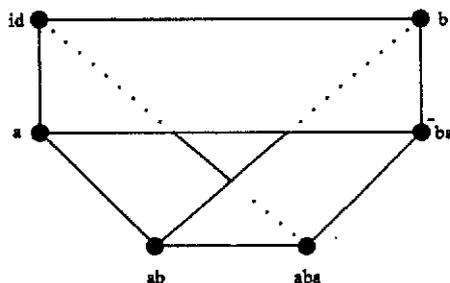


Figure 3. S_3 with a non-standard metric.

S_3 with a non-trivial metric. As the last interesting example we produce the S_3 group with a rather complicated type of metric. Let a and b be the two generators of S_3 , such that $a^2 = b^2 = (ab)^3 = id$. We take $E_a = E_b = E_{aba} = 1$ and that all other coefficients of the metric vanish. Now, we have three nearest neighbours for each point of S_3 . The precise values of distances are rather difficult to calculate and we shall not give these values here. What interests us more is the picture we get by connecting all elements with their nearest neighbours. The object we obtain is presented in figure 3. We easily recognize that its topology (if we look at the rectangular walls) is that of the Möbius strip. This illustrates that the metric on the finite dimensional objects may, in some sense, generate ‘non-trivial topology’ of the resulting lattices.

4. Field theories

Having the metric and the formalism of the differential geometry, we can construct field theories for such spaces. The general procedure and some examples of unitary gauge theories were presented in our earlier work [8]; here we want to concentrate on different aspects of the theory.

Our basic algebra \mathcal{A} is again the commutative algebra of complex-valued functions on the group G . The unitary group of this algebra, $\mathcal{U}(\mathcal{A})$, contains all functions valued in the circle S^1 . The algebra \mathcal{A} is generated by $\mathcal{U}(\mathcal{A})$ or any of its subgroups.

4.1. Discretized target space

To build a physical theory one requires a Hermitian vector bundle over the base space or, equivalently, a projective module over \mathcal{A} . Taking a Hermitian module we may construct the simplest action in the usual way

$$S = \int \langle dm | dm \rangle + V(\langle m | m \rangle) \tag{14}$$

where m are the elements of the module and V is an arbitrary potential function. This approach has been dealt with in many works [1-7].

However, this is not what we now seek, as this would not lead us to theories having a discretized target space. The desired formalism seems to be similar to that of the sigma models where we take the group-valued fields. If we take U to be an element of any group generating the algebra \mathcal{A} , then the proposed expression for the action

$$S = \int \frac{1}{2} \eta (dU^*, dU) + \tilde{V}(U) \tag{15}$$

makes perfect sense. The action (15) is quite natural; it contains both the ‘kinetic’ and the ‘potential’ terms. The latter must be however restricted, so that the value of S is real. From the point of view of field theory the ‘kinetic’ term describes the dynamics of the field and the other term its self-interaction. However, we shall see later, there is another, more intuitive, interpretation.

We shall use this prescription to construct simple models of the discrete geometry. We take the group \mathcal{H} to be any subgroup of $\mathcal{U}(\mathcal{A})$ and the action precisely as defined in (15), with the integration on the algebra being the already introduced Haar integration. Taking into account the form of the metric (11) and the rules of the differential calculus (2)–(8) we may rewrite the action as follows:

$$S = \int_G \left(-\frac{1}{2} \sum_{g \in G'} E_g (U^* - R_g U^*)(U - R_g U) \right) + V(U) . \tag{16}$$

Using the properties of the integration we finally obtain

$$S = \int_G \left(- \sum_{g \in G'} E_g \operatorname{Re} (U^*(R_g U)) \right) + V(U) \tag{17}$$

where we have omitted the constants coming from U^*U terms. Now, we shall attempt to rewrite (17) in a slightly more convenient and recognizable form. Remember that the Haar integration is simply the sum of over all elements of the group G

$$\int_g f = \sum_{h \in G} f(h) \tag{18}$$

so that the potential term splits into the sum of independent contributions from each point of the base space

$$\int_G V = \sum_{h \in G} V(U(h)) . \tag{19}$$

The ‘kinetic’ term is more interesting. Remember that the coefficients E_g define the metric

integration is simply the sum over nearest neighbours with certain weights. Therefore we can rewrite this term as

$$\int_G \sum_g E_g U(R_g U) = \sum_{\substack{h, g \in G \\ \text{nearest neighbours}}} W(g, h) U(h) U(g) \quad (20)$$

where $W(g, h)$ is the weight, which equals $E_{(h^{-1}g)}(h)$.

4.2. Examples

Having constructed the general form of the action we can now present a few interesting examples. We restrict ourselves only to the most distinctive situations as we want only to demonstrate the analogies between the models of statistical physics and of non-commutative geometry.

The Ising model. Let us take the group \mathcal{H} to be the group of \mathbb{Z}_2 -valued functions on \mathbb{Z} . Because for any \mathbb{Z}_2 -valued function $U^2 = 1$, the potential term can be reduced to the linear form $V(U) \sim U$. If we fix the metric to be the standard metric on \mathbb{Z} , as in the first example of the previous section, we get the following action:

$$\mathcal{S} = \sum_{n \in \mathbb{Z}} \alpha U(n) U(n+1) + \gamma U(n) \quad (21)$$

where α, γ are arbitrary real constants. We easily recognize that the action describes the Ising model precisely. Note that the 'kinetic' term of our field theory has now the meaning of the interaction between the nearest neighbours, while the 'potential' term has no other specific interpretation, apart from being the interaction with some external fields. The constant α sets the value of the gap between the energy levels of the model. The path integral is now simply the partition function of the Ising model.

If the potential term is absent the action \mathcal{S} possesses a global symmetry as the change $U \rightarrow -U$ leaves the action invariant.

The Ising model with a non-standard metric. As the next example we take again the same group and the same base space but with a different metric. This time we assume that the metric is as in the second example of the last section, i.e. $E_1 = E_2 = 1$ and that all other coefficients vanish. Then, after similar steps as in the previous case, we obtain the following action:

$$\mathcal{S} = \sum_{n \in \mathbb{Z}} \alpha (U(n)U(n+1) + U(n)U(n+2)) + \gamma U. \quad (22)$$

This again is a variation of the Ising model, although on a slightly modified lattice with each point having four neighbours, as symbolically represented in figure 2.

The three-state Potts model. Consider now the group of \mathbb{Z}_3 -valued functions. Following the same procedure as in the two previous cases we construct the action, taking as the metric over \mathbb{Z} again the standard metric (the same as in the first example). Then the interaction term reads

$$\mathcal{S} = - \sum_n \alpha \operatorname{Re} (U(n)^* U(n+1)). \quad (23)$$

This action (modified slightly by adding an appropriate potential term) can be recognized as the one describing a three-state Potts model. By changing the metric we may, of course, modify the interaction by increasing the number of nearest neighbours.

All these examples deal with one-dimensional models, but the generalization to higher dimensions is straightforward. For instance, one has to take the group \mathbb{Z}^n to obtain the n -dimensional generalization of the considered models. If we want to restrict theories to a finite base space (so that the action is a finite sum), we take the base space to be \mathbb{Z}_N , and take the limit $N \rightarrow \infty$ to recover the case of the \mathbb{Z} -based model.

These examples illustrate that the simple models of statistical physics have their interpretation as a field theory constructed in the framework of non-commutative geometry. They are all built in a rather simple fashion, using the commutative group \mathbb{Z} as the base space and a finite commutative unitary group as the target space. One may, of course, attempt to go beyond that and use the same tools to construct more sophisticated theories, for the non-Abelian groups, for instance. Another new possibility is to construct gauge theories, extending the observed global symmetries to the local ones. We shall see the construction of such an example in the next section.

5. The gauge theory

We shall now briefly outline the prospects of creating the gauge theory by exploiting the symmetry that we have noticed in the last section. The natural extension of the observed global symmetry is the group \mathcal{H} itself, so we propose it as a gauge symmetry group.

Following the construction procedures from our earlier work [8] we take the gauge connection one-form Φ

$$\Phi = \sum_{g \in G'} \Phi_g \chi^g \tag{24}$$

where the coefficients Φ_g belong to the algebra \mathcal{A} . It will be convenient to use the shifted connection, $\Psi_g = 1 - \Phi_g$, as then all the expressions simplify considerably. Since the group is unitary we have the hermicity constraint, which is

$$\Psi_g^* = R_g(\Psi_{(g^{-1})}) . \tag{25}$$

The curvature two-form $F = d\Phi + \Phi\Phi$ expressed in terms of Ψ reads

$$F_{gh} = \Psi_g R_g(\Psi_h) - \Psi_{(h\circ g)} \tag{26}$$

where we identify Ψ_e with 1. Having the metric η of the form (11) we can construct all possible Yang–Mills type actions

$$S_1 = \int_G \sum_{g,h} E_g E_{(h^{-1})} (\Psi_g \Psi_g^* - 1) (\Psi_h^* \Psi_h - 1) \tag{27}$$

$$S_2 = \int_G \sum_{g,h} E_g E_{(h^{-1})} (\Psi_g R_g \Psi_h - \Psi_{(h\circ g)}) (\Psi_g R_g \Psi_h - \Psi_{(h\circ g)})^* \tag{28}$$

$$S_3 = \int_G \sum_{g,h} E_g E_h (\Psi_g R_g \Psi_h - \Psi_{(h\circ g)}) (\Psi_h R_h \Psi_g - \Psi_{(g\circ h)})^* . \tag{29}$$

We shall concentrate now on the particular case of the Ising model. The metric is defined by taking $E_1 = 1$ and $E_g = 0$ for $g \neq 1$. This fixes the actions (27)–(29) to take the following form:

$$S_1 = \sum_n (\Psi_1(n) \Psi_1^*(n) - 1)^2 \tag{30}$$

$$S_2 = S_1 \tag{31}$$

$$S_3 = \sum_n |(\Psi_1(n) \Psi_1(n+1) - \Psi_2(n))|^2 . \tag{32}$$

First, let us note that only the fields Ψ_1 and Ψ_2 contribute to the action, which follows from our choice of the metric. Moreover, this choice causes the action S_1 to have no interaction terms, which causes the value of $\Psi_1(n)$ to be independent of other values of this field for every n . The situation is somehow different in the third possible action, where we have both the interaction term for Ψ_1 and the interaction between Ψ_1 and Ψ_2 .

The model described by the first action (30) is not interesting from the physical point of view, as it describes a completely non-interacting system. We shall not discuss here the other action and its properties, as it does not resemble any model of statistical physics.

6. Conclusions

We have shown a way to construct a class of field theories in discrete geometry, which have discretized target spaces. We found that some of them correspond exactly to well known models of statistical physics. We were able to modify them slightly by changing the free parameters of our construction, which were the metric and the potential.

Let us summarize the most important facts about the construction. The space of fields was chosen to be a subgroup of the unitary group of the algebra \mathcal{A} , which determined the target space. The form of the interaction was dependent only on the metric of the base space and it appeared in the action as a 'kinetic' term. We also allowed a potential term. This determined the action and the model completely.

We believe that the correspondence with field theory, which we presented in this paper for the Ising model and the three-state Potts model, can be extended to many other systems. Moreover, using this method we may be able to analyse and compare their properties from another viewpoint: we may also use the methods to create other models, by fitting the algebra \mathcal{A} , the subgroup \mathcal{H} and the metric η . Whether such models would exhibit any interesting features remains an open question.

Finally, we presented a method of building a gauge theory, using a discretized group of gauge symmetries. It seems, however, that the resulting models, at least in the case studied, were of little physical meaning.

In our considerations we used the commutative algebra \mathcal{A} of complex-valued functions on G . Let us mention here that the same analysis may be repeated for algebras over \mathbb{Z} . In that case the algebra \mathcal{A} would be simply defined as generated by the group \mathcal{H} . We point out that in that case one does not have to restrict oneself to Abelian groups. That situation would be probably the most interesting one.

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